ERRATUM: "Exact Analytical Solutions of Continuity Equation for Electron Beams Precipitating in Coulomb Collisions"

R. R. Dobranskis and V. V. Zharkova
Department of Mathematics and Information Sciences, University of Northumbria, Newcastle upon Tyne, NE1 2XP, UK
valentina.zharkova@northumbria.ac.uk

ABSTRACT

In this erratum letter we correct a mistake in the characteristics for differential density \( N \) obtained in Dobranskis & Zharkova (2014) from the updated continuity equation for electron density and compare the solutions for electron density \( N \) obtained from continuity equations (CEs) for electron flux \( Nv \) and for electron density. We show that the corrected solution for \( N \) obtained from CE for electron density still has an additional exponential term of \((E^2 + 2a\xi)^{-1/4}\) comparing to the solution found from continuity equation for electron flux. This updated solution produces power law differential spectra of beam electrons as function of \( E^2 + 2a\xi \) having a spectral index equal to \( \gamma + 1 \) and not to \( \gamma + 0.5 \) as it appears from the continuity equation for electron flux \( Nv \). This updated solution is exactly the one reported by Syrovatskii & Shmeleva (1972) indicating that their solution has been derived from CE for electron density and not for electron flux as stated in their paper. The difference in the spectral indices in energy spectra is also reflected in the spectral indices for mean electron spectra with a spectral index to be equal to \( \gamma - 2 \), similar to Syrovatskii & Shmeleva (1972), for the solutions from CE for electron density and equal to \( \gamma - 2.5 \) if continuity equation for electron flux is used.

Subject headings: Sun: flares – Sun: particle emission – Sun: X-rays, gamma rays

1. Corrected solutions for differential density

The updated solution of the Continuity equation (CE) presented in Dobranskis & Zharkova (2014) has a mistake, which occurred during the application of method of characteristics when solving the characteristic equation for differential density \( N \).

The corrected application of the method of characteristics for non-linear first order partial differential equation for density \( N \), derived by Dobranskis & Zharkova (2014) (their equation 13):

\[
\frac{\partial N}{\partial \xi} - a \frac{\partial N}{E \partial E} = \frac{aN}{2E^2}.
\]

It is assumed that the values of \( N \) are known on some initial curve (the boundary condition):

\[
\xi(0) = \xi_0, \quad E(0) = E_0, \quad N(0) = N_0,
\]

where \( E \) and \( N \) are given as functions of \( \xi \). Then the characteristic equations for each variable: \( \xi, E \) and function \( N \) are solved in the following way:

\[
\frac{d\xi}{dt} = 1 \quad \rightarrow \quad \xi(0) = 0 \quad \rightarrow \quad \xi = t \ldots (+\xi_0), \tag{3}
\]

\[
\frac{dE}{dt} = -\frac{a}{E} \quad \rightarrow \quad E(0) = E_0
\]

\[
\rightarrow \quad \frac{E^2}{2} = -at + \frac{E_0^2}{2} \quad \rightarrow \quad E_0 = \sqrt{E^2 + 2a\xi}, \tag{4}
\]

\[
\frac{dN}{N} = \frac{1}{2} \frac{a}{E^2} dt \quad \rightarrow \quad \frac{dN}{N} = -\frac{1}{2} \frac{a}{E_0^2} dt - 2at dt. \tag{5}
\]

Hence, the energy \( E \) in the characteristic equation (5) for \( N \) above, is substituted with the expression \( E^2 = E_0^2 - 2at \) obtained from the characteristic equation (4) for \( E \).
The characteristic equation for \( N(5) \) can then be integrated, to produce the solution:

\[
\ln(N(N_0) = \frac{1}{4} \ln(u)|_{u_0}^u \\
\rightarrow N = N_0 \times (u)^{\frac{1}{4}} \times (u_0)^{-\frac{1}{4}}. \quad (6)
\]

Then, after making the relevant substitutions:

\[
u(t) = E_0^2 - 2at, \quad u_0(t = 0) = E_0^2,
\quad (7)
\]

the general solution of the CE is obtained:

\[
N(E, \xi) = \Psi(\sqrt{E^2 + 2a\xi}) \times E^{\frac{1}{2}} \\
\times (E^2 + 2a\xi)^{-\frac{1}{4}}. \quad (8)
\]

The equation (8) still has an additional exponential factor of \((E^2 + 2a\xi)^{-1/4}\) (the last term in Equation (8)) appearing in the continuity equation for \( N \). Yet, the solution of continuity equation for electron flux \( N_v \) is defined by only the first two factors \( \Psi(\sqrt{E^2 + 2a\xi}) \times E^{\frac{1}{2}} \).

Substitution of the initial condition (Equation 2 in Dobranskis & Zharkova (2014)) produces the final solution of CE for \( N \):

\[
N(E, \xi) = KE^{\frac{1}{2}} (E^2 + 2a\xi)^{-\frac{\gamma+0.5}{2}} \\
\times (E^2 + 2a\xi)^{-\frac{1}{4}} \times \Theta\left(\sqrt{E^2 + 2a\xi} - E_{low}\right) \\
\times \Theta\left(E_{upp} - \sqrt{E^2 + 2a\xi}\right). \quad (9)
\]

while the solution for differential density found from continuity equation for electron flux is as follows (Dobranskis & Zharkova 2014):

\[
N(E, \xi) = KE^{\frac{1}{2}} (E^2 + 2a\xi)^{-\frac{\gamma+0.5}{2}} \\
\times \Theta\left(\sqrt{E^2 + 2a\xi} - E_{low}\right) \\
\times \Theta\left(E_{upp} - \sqrt{E^2 + 2a\xi}\right). \quad (10)
\]

It can be seen that, in fact, the solution for differential density \( N \) found from the updated continuity equation (9) is exactly the solution reported by Syrovatskii & Shmeleva (1972), with the index near the term \( E^2 + 2a\xi \) being equal to \(-\frac{\gamma+1}{2}\) and not to \(-\frac{\gamma+0.5}{2}\) as appears from the continuity equation for electron flux \( N_v \). This means that Syrovatskii & Shmeleva (1972) solved the equation for electron density and not for electron flux as stated in their paper and hence, obtained the exact solution for electron density.

In Figure 1 the effects of the updated solutions found from the continuity equation for \( N \) are compared with the results obtained from the original CE for electron beam flux. It can be seen that the updated solutions for differential densities have still a steeper decrease at lower energies than the solutions found from the electron flux equation that confirms the general conclusions of Dobranskis & Zharkova (2014), although making this decrease less pronounced.

2. Mean electron spectra

The mean electron spectra (MES) for the corrected solution (9) are calculated as follows:

\[
\bar{F}(E) = KE \int_0^\infty \left(\frac{2E}{m}\right)^{\frac{\gamma+0.5}{2}} \\
\times E^\frac{1}{2} \times (E^2 + 2a\xi)^{-\frac{\gamma+0.5}{4}} d\xi \\
= KE \sqrt{\frac{2}{m}} \int_0^\infty (E^2 + 2a\xi)^{-\frac{\gamma+1}{4}} d\xi. \quad (11)
\]

The integration produces:

\[
\bar{F}(E) = KE \sqrt{\frac{2}{m}} \frac{E^{2-2n}}{n - 1}, \text{ for } E_{low} \leq E, \quad (12)
\]

where the spectral index \( n \) for the updated solution (9) is defined as:

\[
n = \frac{\gamma + 1}{2}, \quad (13)
\]

while for the original solution it is equal to:

\[
n_1 = \frac{\gamma + 0.5}{2}. \quad (14)
\]

The substitution of \( n \) and \( n_1 \) into the formula (12) for MES results in the following dependence of MES on energy:

for the updated solution (9) from continuity equation (1) for electron density:

\[
\bar{F}(E) = KE \sqrt{\frac{2}{m}} \left(\frac{E^{2-\frac{\gamma+1}{2}}}{\frac{\gamma+1}{2}}\right) \\
= KE \sqrt{\frac{2}{m}} \frac{E^{-(\gamma+1)}}{(\gamma+1)/2 - 1} = CE^{-(\gamma+2)}; \quad (15)
\]
Fig. 1.—: The variation of electron differential density spectra versus energy plotted for column depths of $1.4 \times 10^{20}$ (blue), $1.2 \times 10^{21}$ (green), $4.5 \times 10^{21}$ (magenta), $1.1 \times 10^{22}$ (red) and energy range from 10 keV to 1000 keV. The solid lines represent electron densities $N$ (Eq. 9) found from the continuity equation for electron density and dashed ones - the electron densities (Eq. 10) obtained from continuity equation for electron flux.

for the original solution (10) from continuity equation for electron flux:

$$\bar{F}(E) = \frac{KE}{2a} \sqrt{\frac{2}{m}} E^{2-(\gamma+0.5)}$$

$$= \frac{KE}{2a} \sqrt{\frac{2}{m}} E^{-(\gamma-3/2)} = C_1 E^{-(\gamma-2.5)}. \quad (16)$$

Hence, the MES found from the solution (10) of continuity equation for electron flux in the energy dependence have a spectral index $\gamma-2.5$ that is different from the spectral index $\gamma-2$ found from the updated solutions (9), which are those reported by Syrovatskii & Shmeleva (1972).

The effects of the updated solution on the hard X-ray intensity are compared in Figure 2. Comparison of the solutions of continuity equation for electron flux with those for electron density confirms the conclusion by Dobranskis & Zharkova (2014) that the updated solution (9) (Syrovatskii & Shmeleva 1972) produces a closer fit of the HXR energy spectra to those found from the Fokker-Planck approach than those found from the continuity equation for electron flux.

3. Conclusions

With the correction of a mistake in the characteristics for $N$, the solution (9) for differential density $N$ obtained from the updated continuity equation for electron density still has the additional term of $(E^2 + 2a\xi)^{-1/4}$ comparing to the solution (10) found from continuity equation for electron flux.

After substitution of the boundary conditions, this updated solution produces power law differ-
Fig. 2.—: Hard X-Ray intensity in relative units, found from the original solution (10) of continuity equation for electron flux (red), the updated solution (9) with recovered term (green), the numerical Fokker - Planck - Landau approach (blue). The initial energy fluxes of $10^8$ (top row) and $10^{12}$ erg cm$^{-2}$s$^{-1}$ (bottom row).
Spectral indexes $\gamma = 3$ (left column) and $\gamma = 7$ (right column).

Differential spectra of beam electrons as function of $E^2 + 2a\xi$ having a spectral index equal to $-\frac{2+1}{2}$ and not to $-\frac{2+\gamma}{2}$ as it appears from the continuity equation for electron flux $N_0$. This means that Syrovatskii & Shmeleva (1972) solved the equation for electron density and not for electron flux as stated in their paper and, hence, obtained the exact solution for electron density.

This difference in the spectral indices in energy spectra is also reflected in the spectral indices for mean electron spectra. The use of the updated differential densities found from CE for electron density (Equation (1) here) produces mean electron spectra with a spectral index to be equal to $\gamma - 2$ similar to that reported by Syrovatskii & Shmeleva (1972), while a use of the differential densities derived from continuity equation for electron flux is resulted in the spectral index of mean electron spectra to be equal to $\gamma - 2.5$.

Also, similarly to conclusions by Dobranskis & Zharkova (2014), hard X-ray intensities produced by electrons with the corrected differential energy spectra still fit closer the simulations obtained with Fokker-Planck approach than those obtained from the continuity equation for electron flux.

REFERENCES

This 2-column preprint was prepared with the AAS LaTeX macros v5.2.